MATH 307

HW 6 DUE TODAY
HW 7 DUE NEXT FRIDAY

$\boxed{\text{ENTRY TASK}}$ Using Laplace transforms, solve $y'' + 4y = 8$, $y(0) = 0$, $y'(0) = 3$

$\boxed{I}$ $\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{8\}$

$\boxed{II}$ $s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 4\mathcal{L}\{y\} = 8\mathcal{L}\{1\}$

$$(s^2+4)\mathcal{L}\{y\} - 3 = \frac{8}{s}$$
$$(s^2+4)\mathcal{L}\{y\} = \frac{8}{s} + 3$$
$$\mathcal{L}\{y\} = \underbrace{\frac{8}{s(s^2+4)}} + \frac{3}{s^2+4}$$

$\boxed{III}$ $\dfrac{8}{s(s^2+4)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+4} \Rightarrow 8 = A(s^2+4) + (Bs+C)s$

$s = 0 \Rightarrow A = 2$

$$8 = As^2 + 4A + Bs^2 + Cs$$
$$8 = (A+B)s^2 + Cs + 4A$$

$B = -2$
$C = 0$

$$\begin{cases} A + B = 0 \Rightarrow B = -A \\ C = 0 \\ 4A = 8 \end{cases}$$

$\boxed{IV}$ $\mathcal{L}\{y\} = \dfrac{2}{s} + \dfrac{-2s}{s^2+4} + \dfrac{3}{s^2+4}$ $\left( = \dfrac{2}{s} + \dfrac{-2s+3}{s^2+4} \right)$

$$y(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$\boxed{2 \cdot 1 - \underbrace{2\cos(2t) + 3\,\tfrac{1}{2}\sin(2t)}_{\text{homogeneous sol'n}}}$$

particular sol'n

$$\mathcal{L}\{\sin(bt)\} = \frac{b}{s^2+b^2}$$
$$\sin(bt) = \mathcal{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\}$$
$$\frac{1}{b}\sin(bt) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+b^2}\right\}$$

| LAPLACE TRANSFORM PRO | $ay'' + by' + cy = g(t)$ CON |
|---|---|
| • TURNS INTO PURELY ALGEBRA PROBLEM (NO "GUESSING") | • MESSY ALGEBRA |
| • SOLVES HIGHER ORDER | • ONLY CONSTANT COEFFICIENT (CAN GENERALIZE THOUGH) |
| • CAN HANDLE NONHOMOGENEOUS EQUATIONS SYSTEMATICALLY (DISCONTINUOUS) | • HAVE TO LOOK UP IN TABLE |

**Ex** $y'' + 4y' + 4y = t^4 e^{-2t}, \quad y(0) = 0, \quad y'(0) = 1$

$(At^4 + Bt^3 + Ct^2 + Dt + E)e^{-2t}$

REPEATED ROOT $r = 2 \rightarrow t^2$

I $\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{t^4 e^{-2t}\}$

II $s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 4s\mathcal{L}\{y\} - 4y(0) + 4\mathcal{L}\{y\} = \dfrac{4!}{(s+2)^5}$

$$\underbrace{(s^2+4s+4)}_{(s+2)^2}\mathcal{L}\{y\} - 1 = \frac{4!}{(s+2)^5}$$

$$(s+2)^2\mathcal{L}\{y\} = \frac{24}{(s+2)^5} + 1$$

$$\mathcal{L}\{y\} = \frac{24}{(s+2)^7} + \frac{1}{(s+2)^2}$$

III PARTIAL FRACTIONS? DONE ✓

IV $y(t) = 24\,\mathcal{L}^{-1}\left\{\dfrac{1}{(s+2)^7}\right\} + \mathcal{L}^{-1}\left\{\dfrac{1}{(s+2)^2}\right\}$

$$y(t) = 24\,\frac{1}{6!}t^6 e^{-2t} + \frac{1}{1!}te^{-2t}$$

$$\boxed{y(t) = \left(\frac{1}{30}t^6 + t\right)e^{-2t}}$$

$\mathcal{L}\{t^n e^{at}\} = \dfrac{n!}{(s-a)^{n+1}}$

$t^n e^{at} = \mathcal{L}^{-1}\left\{\dfrac{n!}{(s-a)^{n+1}}\right\}$

$\mathcal{L}^{-1}\left\{\dfrac{1}{(s-a)^{n+1}}\right\} = \dfrac{1}{n!}t^n e^{at}$

$\mathcal{L}^{-1}\left\{\dfrac{1}{(s-a)^n}\right\} = \dfrac{1}{(n-1)!}t^{n-1} e^{at}$

## 6.3 : STEP FUNCTIONS

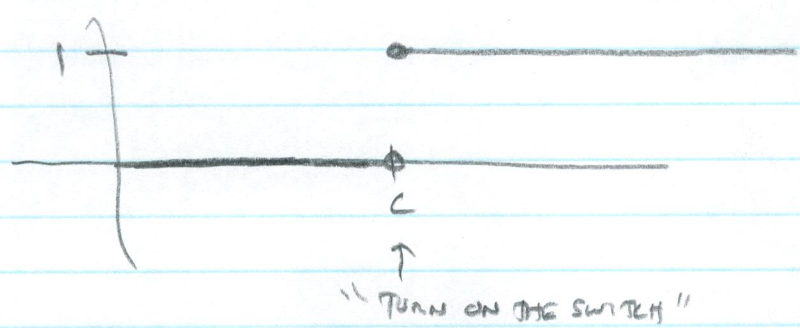
NOW WE CONSIDER PROBLEMS LIKE

$$\left\{ ay''+by'+cy = \begin{cases} 0, & 0\le t<10; \\ 3, & 10\le t<20; \\ \cos(t-20), & 20\le t<30; \\ 0, & 30\le t. \end{cases} \right.$$

TODAY WE LEARN HOW TO ORGANIZE PIECEWISE FORCING IN A USEFUL "ONE-LINE" WAY.

**Def'n** The unit step function (or Heaviside function) is defined by

$$u_c(t) = \begin{cases} 0, & t<c; \\ 1, & t\ge c. \end{cases}$$




**COMBINING**

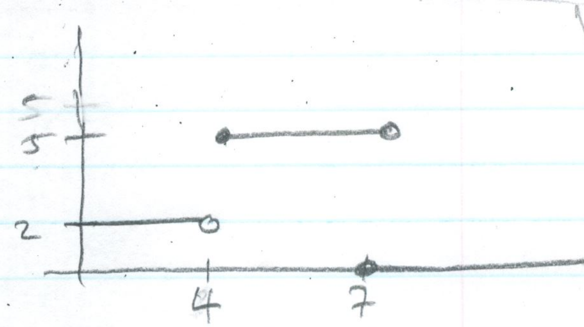
**Ex** What is $f(t) = 2 + 3\underbrace{u_4(t)}_{\begin{cases} 0, & t<4; \\ 1, & t\ge 4, \end{cases}} - 5\underbrace{u_7(t)}_{\begin{cases} 0, & t<7; \\ 1, & t\ge 7. \end{cases}}$ ?

If $t<4$, then $f(t) = 2+3(0)-5(0) = 2$

If $4\le t<7$, then $f(t) = 2+3(1)-5(0) = 5$

If $t\ge 7$, then $f(t) = 2+3(1)-5(1) = 0$

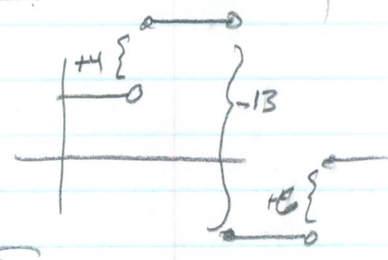
$$f(t) = \begin{cases} 2, & t < 4; \\ 5, & 4 \le t < 7; \\ 0, & t \ge 7. \end{cases}$$




REVERSE THIS

Ex

$$g(t) = \begin{cases} 3, & t < 5; \\ 7, & 5 \le t < 10; \\ -6, & 10 \le t < 15; \\ 0, & t \ge 15. \end{cases}$$




ONE "JUMP" AT A TIME

START = 3    $0 \le t < 5 \Rightarrow g(t) = 3$

"JUMP UP 4 AT $t=5$"    $0 \le t < 10 \Rightarrow g(t) = 3 + 4u_5(t)$

"JUMP DOWN 13 AT $t=10$"    $0 \le t < 15 \Rightarrow g(t) = 3 + 4u_5(t) - 13u_{10}(t)$

"JUMP UP 6 AT $t=15$"    all $t$

$$\boxed{g(t) = 3 + 4u_5(t) - 13u_{10}(t) + 6u_{15}(t)}$$

STOP

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\,dt$$

$$= \underbrace{\int_0^c e^{-st} u_c(t)\,dt}_{0} + \int_c^\infty e^{-st} \underbrace{u_c(t)}_{1}\,dt$$

$$= \int_c^\infty e^{-st}\,dt = \lim_{A\to\infty} \int_c^A e^{-st}\,dt$$

$$= \lim_{A\to\infty} -\frac{1}{s} e^{-st} \Big|_c^A$$

$$= \lim_{A\to\infty} \left( \cancel{-\frac{1}{s} e^{-sA}}^{\,0} + \frac{1}{s} e^{-cs} \right)$$

$$\boxed{\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}} \qquad \boxed{u_c(t) = \mathcal{L}^{-1}\left\{\frac{e^{-cs}}{s}\right\}}$$

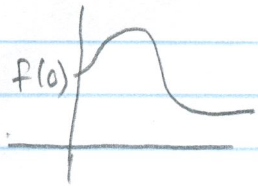
Ex)

$$\mathcal{L}\{3 + 4u_5(t) - 13u_{10}(t) + 6u_{15}(t)\}$$

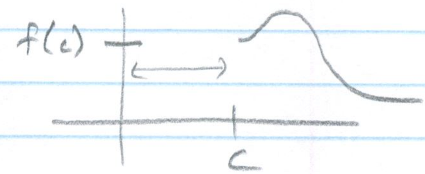
$$= \frac{3}{s} + \frac{4e^{-5s}}{s} - \frac{13e^{-10s}}{s} + \frac{6e^{-15s}}{s}$$

## Shifting

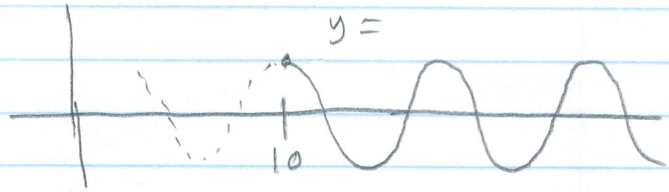
$f(t)$



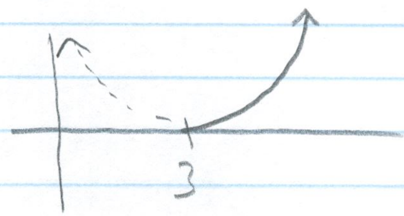

$f(t-c)$




**Ex)** $y = \cos(t-10)$

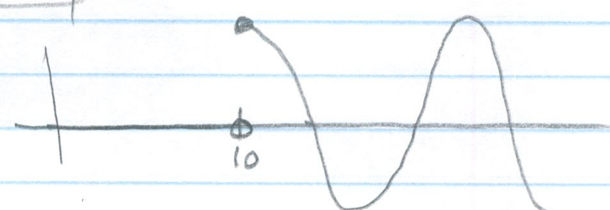



$y = (t-3)^2$

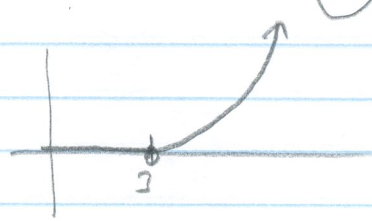



$$\boxed{u_c(t)\, f(t-c) = \begin{cases} 0, & t < c; \\ f(t-c), & t \geq c \end{cases}}$$

$y = u_{10}(t)\cos(t-10)$




$y = u_3(t)\,(t-3)^2$



$$\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty u_c(t)f(t-c)e^{-st}\,dt$$

$$= \int_c^\infty f(t-c)e^{-st}\,dt \qquad \begin{array}{l} u = t-c \\ du = dt \end{array}$$

$$= \int_0^\infty f(u)e^{-s(u+c)}\,du$$

$$= e^{-cs}\underbrace{\int_0^\infty f(u)e^{-su}\,du}_{\mathcal{L}\{f(u)\}}$$

$\boxed{\text{Thm}}$ If $F(s) = \mathcal{L}\{f(u)\}$ (so $\mathcal{L}^{-1}\{F(s)\} = f(u)$)

then $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(u)\} = e^{-cs}F(s)$

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c)$$

$$= u_c(t)\,\mathcal{L}^{-1}\{F(s)\}(t-c)$$ sometimes we may write

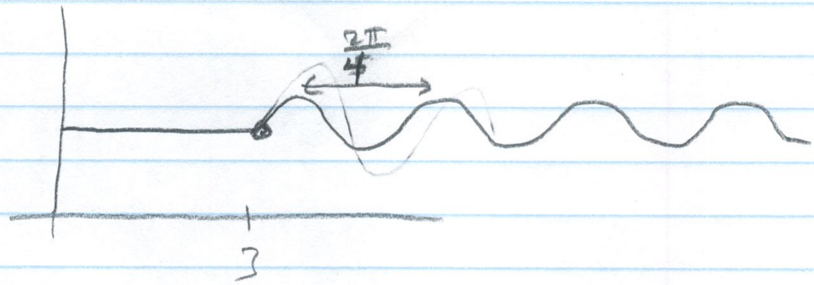
Ex) $\mathcal{L}\{u_{10}(t)\cos(t-10)\} = e^{-10s}\mathcal{L}\{\cos(u)\}$

$$= e^{-10s}\frac{s}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{e^{-7s}\frac{s}{s^2+1}\right\} = u_7(t)\,\underbrace{\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}}_{\cos(u)}(t-7)$$

$$= u_7(t)\cos(t-7)$$

Ex) $$f(t) = \begin{cases} 6 & ; \ 0 \le t < 3 ; \\ 6 + 2\sin(4(t-3)) & ; \ 3 \le t \end{cases}$$




$$\boxed{f(t) = 6 + 2u_3(t)\sin(t-3)}$$

$$\mathcal{L}\{6 + 2u_3(t)\sin(t-3)\}$$

$$\mathcal{L}\{6\} + 2\,\mathcal{L}\{u_3(t)\sin(4(t-3))\}$$

$u = t - 3 \qquad t = u + 3$

$$\frac{6}{s} + 2e^{-3s}\,\underbrace{\mathcal{L}\{\sin(4u)\}}$$

$$\frac{6}{s} + 2e^{-3s}\frac{4}{s^2+16} = \frac{6}{s} + \frac{8e^{-3s}}{s^2+16}$$

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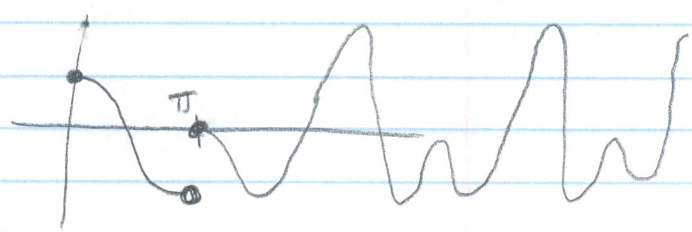
Ex) $$g(t) = \begin{cases} \cos(t) & , \ 0 \le t < \pi ; \\ \cos(t) + \cos(2(t-\pi)) & , \ \pi \le t. \end{cases}$$

$$g(t) = \cos(t) + u_\pi(t)\cos(2(t-\pi))$$




$$\mathcal{L}\{g(t)\} = \frac{s}{s^2+1} + e^{-\pi s}\frac{s}{s^2+4}$$